RG in Momentum space

(Wilson RG)

Above we argued that the Gaussian fixed point in the Ising model (i.e. ϕ^4 theory) must be unstable in $d<4$ due to relevance of interactions, i.e. the u-term. To the leading order at the Gaussian fixed point, we derived,

$$\frac{dt}{dl} = 2t + \ldots$$

$$\frac{dh}{dl} = \left(\frac{d}{2}+1\right) h + \ldots$$

$$\frac{du}{dl} = (4-d)\, u + \ldots$$

Since t and u are both Ising symmetric, we expect that generically, they will mix under RG. Further, our expectation is that the Gaussian fixed point will become unstable to a fixed point with <u>only one</u> thermal direction and <u>only one</u> magnetic field like direction.

The basic idea of the perturbative RG we will now do is that in $d=4$, the $u\varphi^4$ term is marginal at the leading order. Therefore, in $d=4-\varepsilon$ dimension, when ε is small, there is a possibility of finding a fixed point. For example, if the RG flow for u is $\frac{du}{dl} = \varepsilon u - u^2$, then there is a perturbatively accessible fixed point at $u^* = \varepsilon$.

To carry out the RG, let's recall the action (for simplicity we set $h=0$ – this will allow us to only calculate y_t and not y_h. We can redo the calculation with non-zero h once we have understood the methodology).

$$S = \int d^dx \left[\frac{(\nabla \varphi)^2}{2} + \frac{t}{2}\varphi^2 + u\varphi^4 \right]$$

$$= \int_0^{\Lambda} d^dk \left(\frac{k^2+t}{2}\right)|\varphi(k)|^2 + u\int_{k_1k_2k_3k_4} \varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\,\delta(k_1+k_2+k_3+k_4).$$

We now separate φ into slow and fast modes as:

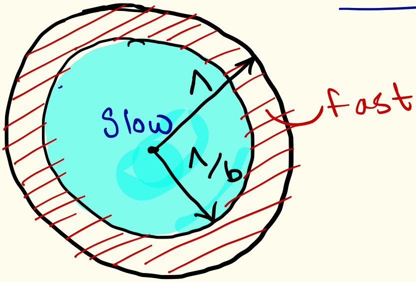
$$\varphi(x) = \underbrace{\int_0^{\Lambda/b} \frac{\varphi(k)e^{ik\cdot x}}{(2\pi)^d}}_{} + \underbrace{\int_{\Lambda/b}^{\Lambda} \frac{\varphi(k)e^{ik\cdot x}}{(2\pi)^d}}_{}$$

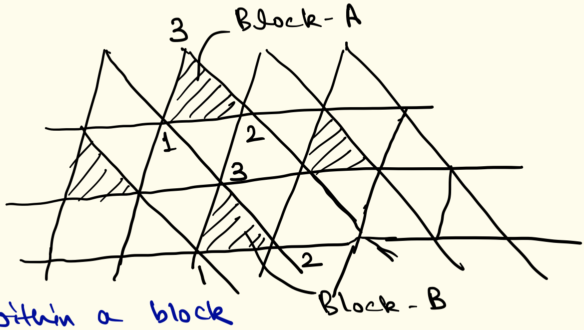
$$= \varphi_<(x) \text{ (“slow”)} + \varphi_>(x) \text{ (“fast”)}$$

$\Rightarrow$ $\varphi_<(k) = \varphi(k)$ if $|k| < \Lambda/b$, and zero otherwise.

$\varphi_>(k) = \varphi(k)$ if $\Lambda/b < |k| < \Lambda$, and zero otherwise.

Analogy with real-space RG:





fast modes ~ spins within a block

slow modes ~ effective (coarse-grained) block spins

The basic idea would be the following:

(i) Integrate out the fast modes to obtain renormalized action for the slow modes.

(ii) Rescale momenta k and field ϕ so that the kinetic energy part of the action is again $\int_0^{\Lambda} d^d k \; \dfrac{|\phi(k)|^2 k^2}{2}$.
This ensures that we are perturbing around the Gaussian fixed point.

(iii) Study how the two relevant couplings at the Gaussian fixed point, namely t and u change as (i) + (ii) is carried out. This leads to the RG flow for t and u.

To illustrate the idea, let us carry out these steps at $u = 0$ to study the flow of t:

$$S_0 = \sum_k \frac{(k^2 + t)}{2V} |\varphi(k)|^2$$

$$= \sum_{|k|<\frac{\Lambda}{b}} \left(\frac{k^2 + t}{2V}\right) |\varphi(k)|^2 + \sum_{|k|=\frac{\Lambda}{b}}^{\Lambda} \left(\frac{k^2 + t}{2V}\right) |\varphi(k)|^2$$

Recall,

$\varphi_<(k) = \varphi(k)$ if $|k| < \Lambda/b$, and zero otherwise.

$\varphi_>(k) = \varphi(k)$ if $\Lambda/b < |k| < \Lambda$, and zero otherwise

$$\Rightarrow S_0 = \underbrace{\sum_k \left(\frac{k^2 + t}{2V}\right) |\varphi_<(k)|^2}_{S_0^<} + \underbrace{\sum_k \left(\frac{k^2 + t}{2V}\right) |\varphi_>(k)|^2}_{S_0^>}$$

Therefore, the slow and fast modes completely decouple for the Gaussian model $\Rightarrow$ integrating out the fast modes only changes the partition fn $Z = \int D\phi_> D\phi_< \, e^{-S}$ only by a pre-factor which we ignore since it doesn't affect any physical quantity.

To implement step (ii), we define $k' = bk$ so that any sum $\sum_{|k|<\Lambda/b} (\ ) = \sum_{|k'|<\Lambda}$.

$$\Rightarrow \quad S_0 = \sum_{|k'|<\Lambda} \frac{k'^2}{b^2} \frac{|\phi(k')|^2}{2V} + \sum_{|k'|<\Lambda} \frac{t}{2V} |\phi(k')|^2$$

Now we restore the part of the action corresponding to the Gaussian fixed point to its original form by defining $\phi'(k') = \dfrac{\phi(k')}{b}$, so that

$$S_0 = \sum_{|k'|<\Lambda} \frac{k'^2 |\phi'(k')|^2}{2V} + t b^2 \sum_{|k'|<\Lambda} \frac{|\phi'(k')|^2}{2}$$

From this, we read off the RG flow of t:

$$t'(b) = t b^2$$

Writing $b = 1 + dl \Rightarrow$

$$\frac{dt}{dl} = 2t$$, thus recovering the RG flow for the Gaussian model.

RG for the interacting ϕ^4 theory:

Now we return to the problem of our main interest, namely the action $S = \int d^d x \left\{ \frac{(\nabla\phi)^2 + t\phi^2}{2} + u\phi^4(x) \right\}$

$$Z = \int D\phi \; e^{-S}$$

$$= \int D\phi_> D\phi_< \; e^{-[S_0(\phi_>) + S_0(\phi_<) + V(\phi_<, \phi_>)]}$$

where $V(\phi_<, \phi_>) = u\int d^d x \; \phi^4(x) = u \int d^d x [\phi_>(x) + \phi_<(x)]^4$

$$\Rightarrow Z = \int D\phi_< e^{-S_0(\phi_<)} \int D\phi_> e^{-S_0(\phi_>)} \; e^{-V(\phi_<, \phi_>)}$$

$$= \int D\phi_< e^{-S_0(\phi_<)} \frac{\int D\phi_> e^{-S_0(\phi_>)} \; e^{-V(\phi_<,\phi_>)}}{\int D\phi_> e^{-S_0(\phi_>)}} \underbrace{\int D\phi_> e^{-S_0(\phi_>)}}_{Z_0^>}$$

$$= \underbrace{Z_0^>}_{\text{we will drop this.}} \int D\phi_< e^{-S_0(\phi_<)} \left\langle e^{-V(\phi_<, \phi_>)} \right\rangle_>$$

where $\langle A \rangle_> = \dfrac{\int D\phi_> A \; e^{-S_0(\phi_>)}}{Z_0^>}$

To exploit the fact that the fixed-point value of u, t are expected to be "small", i.e. $O(\varepsilon)$, we will evaluate $\langle e^{-V(\phi_>, \phi_<)} \rangle_>$ perturbatively, via the cumulant expansion, very similar to the real space RG for the 2d Ising model. This step is what makes the momentum-space RG a practical tool which is also well-controlled, due to small ε.

$$\Rightarrow Z \simeq \int D\phi_< \, e^{-S_0[\phi_<]} \, e^{-\left[\langle V\rangle_> - \frac{[\langle V^2\rangle_> - \langle V\rangle_>^2]}{2} + \dots\right]}$$

Thus, from a technical point of view, the central task is to evaluate expressions such as $\langle V^n\rangle_>$.

Let's consider the two terms, $\langle V\rangle_>$ and $\langle V^2\rangle_>$ separately.

$$\boxed{\langle V\rangle_> = u \int d^dx \left\langle [\phi_>(x) + \phi_<(x)]^4 \right\rangle_>} :$$

We only need to keep track of terms that affect the action of the slow modes, i.e. those involving $\phi_<(x)$.

From $\phi \to -\phi$ sym, only terms $\langle \phi_>^m(x) \rangle$ with m even will be non-zero.

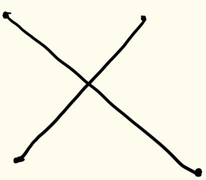
$$u \left\langle \int d^d x \, (\phi_>(x) + \phi_<(x))^4 \right\rangle_>$$

$$= u N \int d^d x \; \phi_<^2(x) \; \langle \phi_>^2(x) \rangle_>$$

$$+ \; u \int \phi_<^4(x) \, d^d x \; + \; \ldots$$

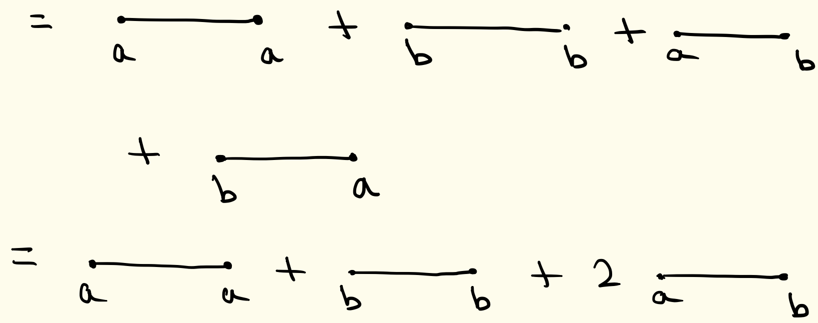
where N is an integer that we need to find, and $\ldots$ includes terms such as $\langle \int \phi_>^4(x) d^d x \rangle$ which just contribute constant prefactors to Z and do not affect the RG flow of $\phi_<$ d.o.f.

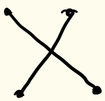
One can find N by explicitly expanding the expression $u \langle \int d^d x \, (\phi_>(x) + \phi_<(x))^4 \rangle_>$. A pictorial way to do it is as follows. One represents the interaction term ϕ^4 as



where each of the four vertices can be either $\phi_>$ or $\phi_<$, and each such possibility corresponds to one of the terms in the expression $[\phi_>(x)+\phi_<(x)]^4$. For example, the equality $(a+b)^2 = a^2+b^2+2ab$ is represented as:

$(a+b)^2 =$



In , there are ${}^4C_2$ ways to select two of the vertices as $\phi_>$, and then the remaining two vertices are assigned $\phi_<$ leaving no additional choice to obtain $\phi_>^2\,\phi_<^2$. Thus,

$N = {}^4C_2 = 6$. Alternatively, one notices that the coefficient of a^2b^2 in $(a+b)^4$ is 6.

Thus, $\langle V\rangle_> = 6u\int d^dx\ \phi_<^2(x)\,\langle\phi_>^2(x)\rangle_> + \int d^dx\,\phi_<^4(x)$

where we have again dropped terms that do not renormalize the action for $\phi_<(x)$ fields.

To evaluate the term $6u \int d^d x \, \phi^2_<(x) \langle \phi^2_>(x) \rangle$

Let's write it in the momentum space :

[recall our fourier transform convention :

$$\varphi(\vec{r}) = \frac{1}{V} \sum_k \varphi(\vec{k}) \, e^{i\vec{k}\cdot\vec{r}}$$

inverse : $$\varphi(\vec{k}) = \int_{\vec{r}} \varphi(\vec{r}) \, e^{-i\vec{k}\cdot\vec{r}}$$]

$$6u \int d^d x \, \phi^2_<(x) \langle \phi^2_>(x) \rangle_>$$

$$= 6u \frac{\int d^d x}{V^4} \sum_{\substack{k_1 k_2 k_3 \\ k_4}} \phi_<(k_1)\, \phi_<(k_2) \, \langle \phi_>(k_3) \, \phi_>(k_4) \rangle_> \; e^{i(k_1+k_2+k_3+k_4)x}$$

using $\int d^d x \, e^{ip\cdot x} = (2\pi)^d \, \delta(p)$, this becomes

$$\frac{6u}{V^4} \sum_{\substack{k_1 k_2 \\ k_3 k_4}} \phi_<(k_1)\, \phi_<(k_2) \, \langle \phi_>(k_3) \, \phi_>(k_4) \rangle \; \delta(k_1+k_2+k_3+k_4)$$

$$= \frac{6u (2\pi)^d}{(2\pi)^{4d}} \int d^d k_1 \, d^d k_2 \, d^d k_3 \, d^d k_4 \; \phi_<(k_1)\, \phi_<(k_2) \, \langle \phi_>(k_3) \, \phi_>(k_4) \rangle_> \; \delta(k_1+k_2+k_3+k_4)$$

Since $\langle \phi_>(k_3)\, \phi_>(k_4) \rangle_>$ expectation value is w.r.t. to the Gaussian action $S_0(\phi^>)$, it is simply given by $\dfrac{(2\pi)^d\, \delta(k_3+k_4)}{k^2+t}$.

$$\Rightarrow \quad 6u \int d^d x \; \phi_<^2(x) \langle \phi_>^2(x) \rangle_>$$

$$= \frac{6u}{(2\pi)^{2d}} \int d^d k \;\; \phi_<(k)\, \phi_<(-k) \int_{\Lambda/b}^{\Lambda} \frac{d^d k_1}{(k_1^2+t)}$$

To remove the clutter coming from the factors of $(2\pi)^d$, let's define $\int đ^d k = \dfrac{\int d^d k}{(2\pi)^d}$, so that

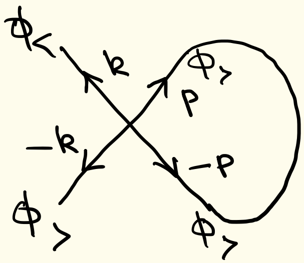
$$6u \int d^d x \; \phi_<^2(x) \langle \phi_>^2(x) \rangle_>$$

$$= 6u \int đ^d k \, |\phi_<(k)|^2 \int_{\Lambda/b}^{\Lambda} \frac{đ^d k_1}{k_1^2+t}$$

Working in momentum space suggests the following pictorial way to calculate various terms in the perturbation theory.

At $O(u^n)$ we draw n interaction vertices

To calculate the contribution to a term that contains m $\phi_<$ fields (e.g. $m=2$ for the quadratic part of the action and $m=4$ for the interaction part), we select m out of the above $4n$ legs and contract the rest of the legs connecting high-energy modes $\phi_>$. Each connecting line gives a contribution of the form $\int_{\Lambda/b}^{\Lambda} \frac{d^d k}{k^2+t}$ where k is the momentum carried by that line. The overall momentum is conserved. For example, at $O(u)$



$$\sim |\phi_<(k)|^2 \int_{\Lambda/b}^{\Lambda} d^d p \, \langle \phi_>(p)\phi_>(-p)\rangle$$

$$\sim |\phi_<(k)|^2 \int_{\Lambda/b}^{\Lambda} \frac{d^d p}{p^2+t}$$

Collecting everything, the renormalized action at $O(u)$ becomes,

S at $O(u)$

$$= S_0[\phi_<] + u\int d^dx \; \phi_<^4(x)$$

$$+ \underbrace{6u\int d^dk \, |\phi_<(k)|^2 \int_{\Lambda/b}^{\Lambda} \frac{d^dk_1}{k_1^2+t}}_{\text{Same form as } S_0[\phi_<] \Rightarrow \text{ renormalizes } t.}$$

where

$$S_0[\phi_<] = \sum_k \frac{k^2+t}{2V} |\phi_<(k)|^2 = \int \frac{d^dk}{(2\pi)^d} \frac{k^2+t}{2} |\phi_<(k)|^2$$

$$= \int d^dk \, \frac{k^2+t}{2} |\phi_<(k)|^2$$

$\Rightarrow$ S at $O(u)$

$$= \int_0^{\Lambda/b} d^dk \, |\phi_<(k)|^2 \left[\frac{k^2+t}{2} + 6u\int_{\Lambda/b}^{\Lambda} \frac{d^dk_1}{k_1^2+t}\right]$$

$$+ u\int d^dx \; \phi_<^4(x)$$

Rescaling : $k' = bk$ $\quad \phi(k) = \dfrac{\phi_<(k)}{b^{1+\frac{d}{2}}}$, so that

S at $O(u)$

$$= \int_0^{\Lambda} d^dk \, |\phi(k)|^2 \left[\frac{k^2}{2} + \frac{t b^2}{2} + 6ub^2 \int_{\Lambda/b}^{\Lambda} \frac{d^dk_1}{k_1^2+t}\right]$$

$$+ u\, b^{4-d} \int d^dx \; \phi^4(x)$$

Therefore, at this order, the RG flow is:

$$t(b) = b^2 \left[t + 12u \int_{\Lambda/b}^{\Lambda} \frac{d^d k}{k^2 + t} \right]$$

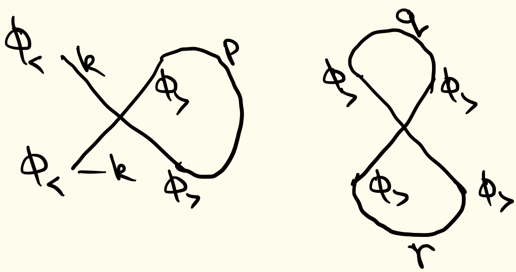
$$u(b) = b^{4-d} u = b^{\varepsilon} u$$

where $t \equiv t(b=1)$, $u \equiv u(b=1)$.

Next, lets consider the contribution from $O(u^2)$ terms.

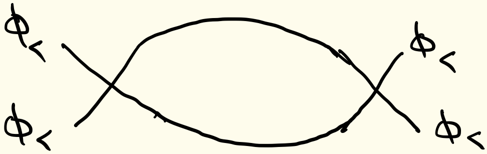
$$\boxed{-\frac{1}{2}\left[\langle V^2 \rangle - \langle V \rangle^2\right]}:$$

The most important simplification is that in any cumulant expansion such as this, only the connected diagrams contribute e.g. the following diagram doesn't contribute to $|\phi_<(k)|^2$



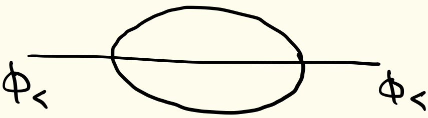
Let's schematically draw potential contributions (most of which will turn out to be unimportant).

(a)

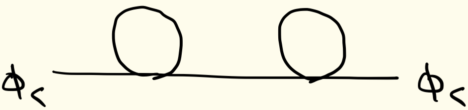


contributes $O(u^2)$ to $\phi_<^4$. $\sim O(\varepsilon^2)$

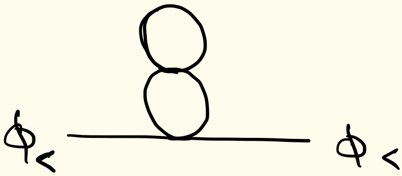
(b)



(c)



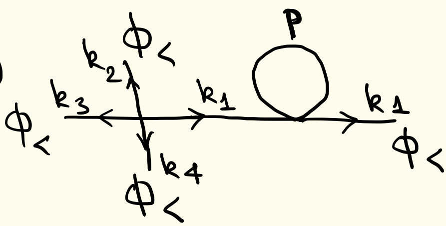
(d)



contributes $O(u^2 t)$ to $\phi_<^2 \sim O(\varepsilon^3)$

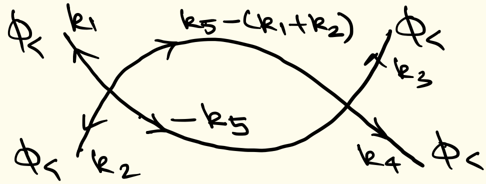
$\Rightarrow$ can be neglected at the accuracy we are working at.

(e)



vanishes since k_1 is an internal leg whose momenta $\in \phi_<$ i.e. k_1 has to satisfy $|k_1| < \Lambda/b$ and $|k_1| > \Lambda/b$ simultaneously.

Therefore, we have to calculate just one term at $O(u^2)$! :



The # of such terms is ${}^4C_2 \times {}^4C_2 \times 2 = 72$

Note that the contribution to the quadratic part at $O(u^2)$ may be neglected at $O(\varepsilon)$. This is because the fixed point values of both t^* and u^* are $O(\varepsilon)$, and the RG performed above at $O(u)$ suffices to give t^* at $O(\varepsilon)$. The contribution from $O(u^2)$ to t^* will be $O(\varepsilon^2)$, which is beyond the accuracy we are calculating at. Therefore, we only need to

Thus, $-\frac{1}{2}\left[\langle V^2\rangle - \langle V\rangle^2\right]$ yields,

$$-\frac{1}{2} \times 72 \times u^2 \times \int_0^{\Lambda/b} đ^d k_1\, đ^d k_2\, đk_3\, đk_4\, \delta\left(\sum_{i=1}^4 k_i\right) \phi_<(k_1)\,\phi_<(k_2)\,\phi_<(k_3)\,\phi_<(k_4)$$

$$\times \int \frac{đ^d k_5}{(t+k_5^2)\,(t+k_1+k_2-k_5)^2}$$

The integral over k_5 involves an integrand that depends on the momenta of external legs.

It turns out this dependence is unimportant.

Heuristically, if one drops the dependence altogether, it leads to a correction to $\phi^4_<(x)$, which is what we are interested in. Keeping the momentum dependence on k_1, k_2 will generate additional terms of the form $\nabla^2 \phi^4_<(x)$ which are anyway irrelevant. Thus, we drop this dependence.

Thus, the above term becomes,

$$-36\, u^2 \int_0^{\Lambda/b} \prod d^d k_i \; \delta\Big(\sum_i k_i\Big)\, \phi_<(k_1)\, \phi_<(k_2)\, \phi_<(k_3)\, \phi_<(k_4) \int_{\Lambda/b}^{\Lambda} \frac{d^d p}{(t+p^2)^2} \, .$$

Combining all terms, both at $O(u)$ and $O(u^2)$, the renormalized action is:

$S =$

$$\int_0^{\Lambda/b} đ^d k \; |\phi_<(k)|^2 \left[\frac{k^2+t}{2} + 6u \int_{\Lambda/b}^{\Lambda} \frac{đ^d k_1}{k_1^2+t} \right]$$

$$+ \int_0^{\Lambda/b} \prod đ^d k_i \; \delta(\textstyle\sum_i k_i) \; \phi_<(k_1)\, \phi_<(k_2)\, \phi_<(k_3)\, \phi_<(k_4)$$

$$\left[u - 36u^2 \int_{\Lambda/b}^{\Lambda} \frac{đ^d p}{(t+p^2)^2} \right]$$

Again, we rescale the momenta and fields as

$k' = bk \qquad \phi(k) = \dfrac{\phi_<(k)}{b^{1+\frac{d}{2}}}$, so that

$$S = \int_0^{\Lambda} đ^d k \; |\phi(k)|^2 \left(\frac{k^2}{2} + \frac{tb^2}{2} + 6ub^2 \int_{\Lambda/b}^{\Lambda} \frac{đ^d p}{p^2+t} \right)$$

$$+ \int_0^{\Lambda} \prod đ^d k_i \; \delta(\textstyle\sum_i k_i) \; \phi(k_1)\, \phi(k_2)\, \phi(k_3)\, \phi(k_4)$$

$$b^{\varepsilon} \left[u - 36u^2 \int_{\Lambda/b}^{\Lambda} \frac{đ^d p}{(t+p^2)^2} \right]$$

Before we proceed, it's worth emphasising that the rescaling $\phi(k) = \dfrac{\phi_<(k)}{b^{1+d/2}}$ works only

because the critical exponent $\eta = 0$ at this order in RG. Suppose for a moment that one recieves a contribution from RG of the form

$$\eta \int_0^{\Lambda/b} đ^d k \, \frac{|\phi_<(k)|^2}{2} k^2 \log(\Lambda/k)$$ Then one may

combine this with the term $\displaystyle\int_0^{\Lambda/b} đ^d k \, |\phi_<(k)|^2 \frac{k^2}{2}$

approximately as $$\int_0^{\Lambda/b} đ^d k \, \frac{|\phi_<(k)|^2}{2} k^{2-\eta}.$$

Now rescaling $k' = bk$, one requires

$$\phi(k) = \frac{\phi_<(k)}{b^{1+\frac{d}{2}-\frac{\eta}{2}}}$$ so that after

rescaling, one obtains $$\int_0^{\Lambda} đ^d k \, \frac{|\phi(k)|^2}{2} k^{2-\eta}.$$

In the φ^4-theory, however, $\eta = 0$ at $O(\varepsilon)$.

Thus, we finally have the RG eqns. we were after,

$$t(b) = b^2 \left[t + 12 u I_1 \right]$$

$$u(b) = b^{\varepsilon} \left[u - 36 u^2 I_2 \right]$$

where $I_1 = \displaystyle\int_{\Lambda/b}^{\Lambda} \frac{đ^d p}{p^2+t}$, $I_2 = \displaystyle\int_{\Lambda/b}^{\Lambda} \frac{đ^d p}{(p^2+t)^2}$

We need the value of these integrals only for $b = 1 + dl$, $d = 4 - \varepsilon$ (ε 'small'), and t also small ($= O(\varepsilon)$).

$$I_1 = \frac{1}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} \frac{d^d p}{p^2+t} = \frac{1}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} \frac{d^d p}{p^2 [1 + t/p^2]}$$

$$\simeq \frac{1}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} \frac{d^d p}{p^2} \left[1 - \frac{t}{p^2} \right] = \frac{1}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} \frac{d^d p}{p^2} - \frac{1}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} \frac{d^d p \, t}{p^4}$$

$$\simeq \frac{S_4}{(2\pi)^4} \int_{\Lambda/b}^{\Lambda} p \, dp - \frac{S_4}{(2\pi)^4} t \int_{\Lambda/b}^{\Lambda} \frac{dp}{p}$$ (S_4 = area of 4-sphere)

$$\simeq K_4 \Lambda^2 \left[1 - \frac{1}{b} \right] - K_4 t \left(1 - \frac{1}{b} \right)$$ $\left(K_4 = \frac{S_4}{(2\pi)^4} \right)$

$$\simeq \quad K_4 \, dl \, [\Lambda^2 - t]$$

Similarly,

$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(k^2+t)^2} = \frac{1}{(2\pi)^d} \int \frac{d^d k}{(k^2+t)^2}$$

$$= \frac{1}{(2\pi)^d} \int \frac{d^d k}{k^4 \left[1 + \frac{t}{k^2}\right]^2}$$

keeping terms only to $O(\varepsilon^0)$ since I_2 multiplies $u^2 = O(\varepsilon^2)$,

$$I_2 = K_4 \, dl$$

Thus, the RG eqns. in the differential form become,

$$t(1+dl) = (1+2dl)\left[t(1) + 12\, u(1) K_4 \, dl \, (\Lambda^2 - t(1))\right]$$

$$u(1+dl) = (1+\varepsilon dl)\left[u(1) - 36\, u^2(1) K_4 \, dl\right]$$

$$\Rightarrow \quad \frac{dt}{dl} = 2t + 12u(\Lambda^2 - t) K_4$$

$$\frac{du}{dl} = \varepsilon u - 36 u^2 K_4$$

We find two fixed points :

(i) Gaussian fixed point at $t^* = u^* = 0$

(ii) A new fixed point (Wilson-Fisher fixed point) at

$$u^* = \frac{\varepsilon}{36 K_4} \; , \quad t^* = -6u^* K_4 \Lambda^2 = -\frac{\Lambda^2 \varepsilon}{6}$$

As we had originally guessed, $u^* = \mathcal{O}(\varepsilon)$ i.e. we are able to access a non-Gaussian fixed point perturbatively in ε.

Linearizing around the Gaussian fixed point

$$\frac{d\,\delta t}{dl} = 2\,\delta t + 12\Lambda^2 K_4\,\delta u$$

$$\frac{d\,\delta u}{dl} = \varepsilon\,\delta u$$

Thus, both t and u directions are relevant, which we anyway knew.

Linearizing around the Wilson-Fisher fixed point,

$$\frac{d\,\delta t}{dl} = 2[t^* + \delta t] + 12(u^* + \delta u) K_4 [\Lambda^2 - (t^* + \delta t)]$$

$$\frac{d\,\delta u}{dl} = \varepsilon(u^* + \delta u) - 36(u^* + \delta u)^2 K_4 .$$

Keeping terms only to linear order in δt, δu and $O(\varepsilon)$,

$$\frac{d\,\delta t}{dl} = 2\,\delta t - 12\,u^* K_4\,\delta t + 12\,\delta u K_4 \Lambda^2 - 12\,\delta u K_4\, t^*$$

$$= \left(2 - \frac{\varepsilon}{3}\right)\delta t + 12\,\delta u K_4 \Lambda^2 \left[1 + \frac{\varepsilon}{6}\right]$$

$$\frac{d\,\delta u}{dl} = \varepsilon\,\delta u - 72\,u^*\,\delta u\, K_4$$

$$= -\,\varepsilon\,\delta u$$

$\Rightarrow$ δu direction is irrelevant and δu flows to zero. $\Rightarrow$ W-F fixed point is the stable one with only one relevant thermal direction.

$$\Rightarrow \frac{d\,\delta t}{dl} = \left(2 - \frac{\varepsilon}{3}\right)\delta t = y_t\,\delta t$$

$$\Rightarrow y_t = \left(2 - \frac{\varepsilon}{3}\right)$$

Furthermore, $t^* = -\dfrac{\Lambda^2 \varepsilon}{6} < 0$ indicates that T_c goes down due to fluctuations.

Of course, t^* and u^* themselves are not universal, unlike y_t and y_u.

One may also check that y_h acquires no correction (homework) compared to Gaussian fixed point : $y_h = \left(1 + \frac{d}{2}\right)$. The simplest way to see this is that the kinetic energy term did not acquire any logarithmic correction

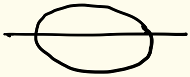
$$\Rightarrow \quad \text{exponent} \quad \eta = 0 \quad \Rightarrow \quad y_h = 1 + d/2 .$$

Using these eigenvalues, we can now calculate approximate exponents for 3d Ising by setting $\varepsilon = 1$!

for example, $\nu \simeq \frac{1}{y_t} = \frac{1}{2\left(1-\frac{\varepsilon}{6}\right)} \simeq \frac{1}{2} + \frac{\varepsilon}{12}$

$$\simeq \frac{1}{2} + \frac{1}{12} \simeq 0.583 .$$

Compare this to the actual value (obtained numerically), $\nu \simeq 0.63$, and the mean-field result $\nu_{MF} = 0.5$. By systematically going to higher order in ε, one can improve upon this, and obtain very accurate results. For explain, at $O(\varepsilon^5)$, one obtains $\nu \simeq 0.63$.

Another point to note is that at $O(\varepsilon)$, $\eta = 0$. This is not a bad approximation, since actual $\eta \simeq 0.04$ for 3d Ising. The smallness of η justifies ε-expansion. The first non-zero correction to η comes at $O(\varepsilon^2)$ from the diagram  and equals $\eta = \varepsilon^2/54 \simeq 0.02$ when $\varepsilon = 1$. Again, by going to higher orders, one obtains excellent agreement e.g. at $O(\varepsilon^5)$, $\eta \simeq 0.037$. See the review article by Wilso and Kogut (Physics Reports 1974) for more details.